THE
MATHEMATICAL GAZETTE.

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ON THE DEVELOPMENT OF MATHEMATICAL ANALYSIS AND ITS RELATIONS TO SOME OTHER SCIENCES.

IV.

Without restricting ourselves to the historical order, let us resume the development of mathematical physics in the last century, in so far as analysis is concerned. The problems of thermal equilibrium lead to the equation already known to Laplace in the study of attraction. There are few equations which have been the object of so many researches as this celebrated one. The conditions for the limits may assume various forms. The simplest case is that of the thermal equilibrium of a body, the elements of the surface of which are maintained at given temperatures. From the physical point of view, it may be regarded as evident that the temperature, assumed continuous in the interior since there is no source of heat, is determined when it is given at the surface. The more general case is that in which, the condition remaining permanent, there would be a radiation outward with an intensity varying at the surface according to a given law; in particular the temperature may be given over one portion, while there is radiation over the remainder. These questions, which are not yet solved in their widest generality, have largely contributed to the direction taken by the theory of partial differential equations. They have called attention to types of determination of the integrals which would never have presented themselves if we had been restricted to a purely abstract point of view. Laplace's equation has been already met with in hydrodynamics, and in the study of attraction varying inversely as the square of the distance. The latter theory brought to light elements of the most essential nature, such as the potential of single and double layers. Here

we meet with analytical combinations of the highest importance, which have since been notably generalised. Green's formula is a case in point. The fundamental problems of electrostatics are of the same order of ideas, and certainly the celebrated theorem on electrical phenomena in the interior of a hollow conductor, which Faraday rediscovered at a later stage by experimental means, knowing nothing whatever of Green's memoir, was a notable triumph for theory. This magnificent aggregate has remained the type of the classical theories of mathematical physics, which seem to us to have almost attained perfection, and which have exercised, and still exercise, so happy an influence on the progress of pure analysis by suggesting to it the most beautiful problems. The theory of functions again will afford us a notable comparison. The analytical transformations brought into play are not distinct from those we have met with in the steady movement of heat. Certain fundamental problems in the theory of functions of a complex variable have lost their abstract enunciation and assumed a physical form, as in the case of the distribution of temperature on a closed surface of any connectivity whatever and without radiation, in thermal equilibrium, with two sources of heat which necessarily correspond to equal and opposite flows. Interpreting this, we find a question on Abelian integrals of the third species in the theory of algebraical curves.

The preceding examples, in which we have considered only the equations of heat and attraction, show that the influence of physical theories is exercised, not only on the general nature of the problems which have to be solved, but even in the details of analytical transformations. As an instance in point in recent memoirs on partial differential equations, we find under the name of Green's formula, a formula suggested by the original formula of the English physicist. The theory of electrodynamics and that of magnetism, in the hands of Ampère and Gauss, have also led to important progress. The study of curvilinear integrals and that of surface integrals owe their development to this theory, and formulas such as that of Stokes, which might also be called Ampère's formula, appeared for the first time in physical memoirs. The equations of the propagation of electricity, with which are connected the names of Ohm and Kirchoff, while presenting a marked analogy with those of heat, often offer slightly different limiting conditions; and we know all that cable telegraphy owes to the detailed discussion of the integrals of one of the Fourier equations applied to electricity. The equations long formulated in hydrodynamics, the equations in the theory of elasticity, and those of Maxwell and Hertz in electro-magnetism, afforded problems analogous to those mentioned above, but under still more varied conditions. There

we meet with many as yet unsurmounted difficulties, but what magnificent results do we owe to the study of particular cases, the number of which we should like to see increased! There should also be noted, as concerning analysis and physics alike, the profound differences presented by propagation by methods varying according to the phenomena that are studied. In equations such as that of sound, we have propagation by waves. In the equations of heat every variation is felt instantaneously at any distance, but very slightly at a very great distance, and here we cannot speak of velocity of propagation. In other cases, of which Kirchoff's equation relative to the propagation of electricity with induction and capacity affords the simplest type, there is a wave front with a given speed, but with a residue behind which does not vanish. These different circumstances reveal very different properties of integrals; research into them has only taken place in a few particular cases, and it raises problems in which we have to deal with the profoundest ideas in modern analysis.

V.

I shall now enter into certain analytical details which are especially interesting in mathematical physics. The question of the generality of the solution of a partial differential equation presents what appear to be paradoxes. For one and the same equation the number of arbitrary functions figuring in the general integral is not always the same, but depends on the form of the integral in question. Thus Fourier, discussing the equation of heat in an unlimited medium, regards it as evident that a solution will be determined if we are given its value for $t=0$, *i.e.*, if we are given *one* arbitrary function of the three co-ordinates, x, y, z. From Cauchy's point of view we may consider, on the contrary, that there are in the general solution two arbitrary functions of three variables. In reality, the question in the form in which it has long been set has no precise signification. In the first place, when it is only a question of analytical functions, any finite number of functions whatever, with any number whatever of independent variables, does not present from the arithmetical point of view more generality than a single function of a single variable, since in both cases the aggregate of the coefficients of the developments forms an enumerable sequence. But there is something more. In reality, in addition to the conditions which are represented by given functions, an integral is subject to conditions of continuity, or must become infinite in a given manner for certain elements. We may thus be led to regard the condition of continuity in a given space as equivalent to an arbitrary function. And so we see how the question of the enumeration of the arbitrary functions is improperly formulated. It is sometimes a delicate matter to prove that

conditions determine a solution uniquely, when we are unwilling to rest content with probabilities; in these cases we must state in exact terms the manner in which the function and certain of its derivatives behave. Thus, in Fourier's problem referring to an unlimited medium, certain hypotheses should be made as to the function and its derivatives at infinity, if we wish to establish that the solution is unique.

Formulas analogous to those of Green are very useful, but the proofs deduced from them are not always entirely rigorous, implicitly assuming conditions fulfilled for the limits which *à priori*, at least, are not necessary. This is one among several other instances of the evolution of exigency in the rigour of proofs. Let me also remark that the new investigation which is now necessary has often led to our being the better able to understand the nature of integrals; this is because true rigour is fertile, thereby being distinguished from rigour of another type, which is purely formal and distasteful, and which casts a shadow over the problems it touches.

Difficulties in the proof of the uniqueness of a solution may be very different, according as it is a question of equations of which all the integrals are or are not analytical. This is an important point, and one that shows that, even although one would like to avoid them, we must sometimes take into account functions that are not analytical. For instance, it cannot be affirmed that Cauchy's problem determines uniquely a solution, when the data of the problem are general—that is to say, when not characteristic. This is surely the case if we consider only analytical integrals, but in the case of non-analytical integrals there may be contacts of an infinite order. And here theory does not outstrip applications; the contrary is the case, as the following example will show. Does Lagrange's celebrated theorem on the velocity-potential in a perfect fluid hold good in a viscous fluid? Examples have been given[1] in which the co-ordinates of the different points of a viscous fluid, starting from rest, cannot be expressed in analytical functions of the time which has elapsed since the initial moment of the motion, and in which the rotations, as well as all their derivatives with respect to the time at that moment are null, and yet are not identically null; Lagrange's theorem therefore does not hold. These considerations sufficiently show the interest that may attach to our ascertaining whether all the integrals of a system of partial differential equations continuous, as well as all their derivatives, up to a given order in a certain field of real variables are analytical functions; it is understood that we assume there are only analytical elements in the equations. For linear equations we have precise theorems,

[1] In a Memoir by M. Boussinesq (*Comptes rendus*, Mar. 29 and Ap. 26, 1880).

all the integrals being analytical, if the characteristics are imaginary, and very general propositions have also been obtained in other cases. And let me remark again that the conditions for the limits which one has to assume are widely divergent, according as the integrals of the equation with which we are concerned are or are not analytical. A question of the first type is given by the generalised problem of Dirichlet. Conditions of continuity play in it an essential part, and in general the solution cannot be extended on both sides of the continuum with reference to which the data are defined. This is no longer so in the second case, in which the relation of this continuum with respect to the characteristics plays the principal part, and in which the field of existence of the solution is presented under quite different conditions. All these ideas, which are difficult to express accurately in ordinary language, are fundamental in mathematical physics, and are not of less interest in infinitesimal geometry. It will suffice to remember that all surfaces of constant positive curvature are analytical, while non-analytical surfaces of constant negative curvature exist.

From the earliest times a vague belief in a certain economy in natural phenomena has been manifest. One of the first detailed examples is given by Fermat's principle relative to the economy of time in the transmission of light. It was next recognised that the general equations of mechanics correspond to a problem of minimum, or, more exactly, to a problem of variation; and thus were derived the principle of virtual velocities, and, later, Hamilton's principle, and the principle of least action. A large number of problems then appear to correspond to the minima of certain definite integrals. This was a very important advance, for the existence of a minimum in many cases could be regarded as evident, and therefore the proof of the existence of a solution was effected. This method of reasoning rendered vast services. The greatest geometers—Gauss, in the problem of the distribution of an attracting mass according to a given potential; Riemann, in his theory of Abelian functions—were satisfied with it. Our attention has been recently drawn to the danger of proofs of this kind; it may be that the minima are simply limits, and that they cannot be effectively attained by real functions possessing the necessary properties of continuity. We are therefore no longer content with the plausibilities which are afforded by the form of reasoning which was classical for so long. Whether we proceed indirectly, or whether we try to give a direct proof of the existence of a function corresponding to the minimum, our way is long and arduous. But it will be none the less useful in every case to connect a question in mechanics or mathematical physics with a problem of minimum. In this there is from the outset a fertile source of analytical transformations,

and besides, in the very processes of the investigation of variations, useful indications may appear relative to the conditions for the limits. A fine example has been given of this by Kirchoff, in his delicate investigation of the conditions for the limits of the equilibrium of flexure of plates.

VI.

I have been led to extend my remarks in particular on partial differential equations. Examples selected from theoretical mechanics and from celestial mechanics would readily show the part that is played by ordinary differential equations in the progress of these sciences, the history of which, as we have seen, has been so closely connected with that of analysis. When the hope of integrating by means of simple functions was abandoned, we were compelled to find developments enabling us to follow up a phenomenon as long as possible, or at any rate to obtain information as to its qualitative aspect. Practically, methods of approximation play an extremely important part in mathematics, and it is by their means that the most advanced parts of theoretical arithmetic come into connection with the applied sciences. As for series, the very existence-proofs of the integrals furnish them from the outset. Thus Cauchy's first method gives developments which are convergent so long as the integrals and the differential coefficients remain continuous. When circumstances enable us to foresee that this will always be the case, the developments obtained are always convergent. In the problem of the n bodies we can in this manner obtain some developments which are valid as long as there are no impacts. If the bodies, instead of attracting, repel one another, this fact need not dismay us, and we obtain developments which are valid indefinitely. Unfortunately, as Fresnel said one day to Laplace, "Nature is not disturbed by analytical difficulties," and the celestial bodies attract and do not repel. In the same way one would be tempted at times to go even farther than the great physicist, and to say that nature has sown the path of the analyst with difficulties. Thus to take another example, if we are given a system of differential equations of the first order, we can generally decide if the general solution is stable about a point or not, and we can find developments in series valid for stable solutions; the sole condition is that certain inequalities must be verified. But if we apply these results to the equations of dynamics in the discussion of stability we find ourselves exactly in the one particular unfavourable case. In general, even here it is not possible to come to a conclusion as to stability. In the case of a force-function having a maximum, the classical, though indirect, method of reasoning establishes the stability which cannot be deduced from any development valid for every value of the time. Do not let us be dismayed by these

difficulties. They will be the source of future progress. Such are also the difficulties that, in spite of the attention that has been paid to them, meet us in the equations of celestial mechanics. From the days of Newton astronomers have drawn from them, by means of practically convergent series and skilfully devised approximations, almost all that is necessary for the prediction of the movements of the heavenly bodies. Analysts would ask more, but little hope is left of reaching the integration by means of simple functions or of developments always convergent. What the admirable researches of modern times have taught them most is the immense difficulty of the problem. A new way has, however, been opened by the study of particular solutions, such as the periodic solutions and the asymptotic solutions which have already been utilised. It is not so much perhaps on account of practical requirements as for the fear of confessing itself vanquished that analysis would never resign itself to abandon without a decisive victory a subject in which it has met with so many brilliant triumphs; besides, what finer field could be found by the budding or rejuvenated theories of the modern doctrine of functions in which to try their strength than in this classical problem of the n bodies?

It is a delight to the analyst in his applications of equations that he can integrate to meet with known functions, with transcendents already classed. Such meetings are unfortunately rare. The problem of the pendulum, the classical cases of the motion of a solid body around a fixed point, are simple instances in which integration has been effected by means of elliptic functions. It would be also extremely interesting to meet with a question in mechanics which might be the starting point of an important discovery in the theory of functions, such as the discovery of a new transcendental enjoying some remarkable property. I should find some difficulty in giving an instance without having to go back to the pendulum and the origin of the theory of elliptic functions. The interpenetration between theory and application is here much less than was the case in questions of mathematical physics mentioned above. Thus is it explained how in the last 40 years researches on the ordinary differential equations connected with analytical functions have in a large measure an entirely abstract theoretical character. Pure theory is here notably in advance. I have had occasion to observe that it is a good thing that it is so, but here the question is evidently one of proportion, and we may hope to see the old problems profit by the progress accomplished in pure theory. There is no difficulty in giving examples, and I will merely remind you of those linear differential equations involving arbitrary parameters, the singular values of which are roots of transcendental integral functions, and one of which in particular brings into

correspondence the successive harmonics of vibrating membranes with the poles of a meromorphic function.

It happens also that the theory may be an element of classification by leading us to seek for the conditions under which the solution is of a given type, as, for example, when the integral is a uniform function. There have been, and there will be again, many interesting discoveries along this line. The case of the motion of a heavy solid body treated by Mme. de Kowalewski, in which Abelian functions were utilised, has already become a classical instance.

VII.

While studying the reciprocal relations of analysis, mechanics and mathematical physics, we have on our way more than once come into contact with the infinitesimal geometry to which so many celebrated problems are due. In many difficult questions the happy combination of calculus and synthetic reasoning has realised considerable progress, as may be seen in the theories of applicable surfaces and of triple orthogonal systems. There is another part of geometry which plays a great rôle in certain analytical researches—I mean the geometry of situation or *analysis situs*. We know how Riemann has from this point of view made a complete study of the continuum of two dimensions, on which he bases his theory of algebraical functions and their integrals. As the number of dimensions increases, the question of *analysis situs* necessarily become complicated. Geometrical intuition ceases, and the study becomes purely analytical, the mind being guided solely by analogies which may be misleading and must be closely investigated. The theory of algebraical functions of two variables, which carries us into the space of four dimensions, without deriving from *analysis situs* such fruitful aid as has been derived by the theory of the functions of one variable, nevertheless owes to it some useful orientations. Again, there is another order of questions in which the geometry of situation intervenes. In the study of curves traced on a surface and defined by differential equations, the connectivity of this surface plays an important part, and this is notably the case in dealing with geodesic lines. The question of connectivity also occurred long since in analysis, when the study of electric currents and magnetism led to non-uniform potentials. In a more general manner certain multiform integrals of some partial differential equations are met with in difficult theories such as that of diffraction, and there is room for various researches in this direction.

From a different point of view I may here recall the relations of algebraical analysis with geometry, which are so elegantly manifested in the theory of groups of finite order.

A regular polyhedron, such as an icosahedron, is on the one hand the solid familiar to us all. Also, to the analyst, it is a group of finite order corresponding to the different ways in which the polyhedron can be made to coincide with itself. The search for all the types of groups of movements of finite order interests not only geometers but also crystallographers. It amounts essentially to the study of groups of ternary linear substitutions with determinant ± 1, and leads to the crystallographers' 32 classes of symmetry for the complex particle of a Bravais lattice. The packing of similar polyhedra in such a way as to fill space completely exhausts every possibility that may arise in the investigation of the structure of crystals. Ever since the idea of the group was introduced into algebra by Galois it has been considerably developed in various ways, so that we now meet it in every branch of mathematics. In its applications especially it appears to us as an admirable instrument of classification. Whether it be a question of substitution groups or of Sophus Lie's transformation groups, whether it be a question of algebraical or of differential equations, this doctrine, so comprehensive in its scope, enables us to deal with the degree of difficulty of the problems treated, and teaches us to utilise the special circumstances that are presented. On this ground it must prove as valuable in mechanics and mathematical physics as in pure analysis. The degree of development of mechanics and physics has enabled us to give to almost all their theories a mathematical form. Certain hypotheses and a knowledge of the elementary laws have led to differential relations which constitute the final form under which these theories, at any rate for a time, are fixed. Little by little they have seen their field enlarge with the principles of thermodynamics. Chemistry, too, tends nowadays to assume a mathematical form. I shall only refer as a case in point to the celebrated memoir of Gibbs on the equilibrium of chemical systems, so analytical in its character, in which chemists could not without an effort recognise in their algebraical garb laws of considerable importance. It seems as if chemistry has nowadays emerged from the pre-mathematical method with which every science begins, and that the day must come in which will be systematised theories of vast importance and magnitude, analogous to those of our present mathematical physics, but much wider in scope and comprising all physico-chemical phenomena. It would be premature to ask if analysis will find in their developments the source of further progress. We cannot even conjecture the analytical types with which we shall be confronted. I have repeatedly mentioned the differential equations which regulate phenomena. Will this always be the final form into which a theory is crystallised? On this point, indeed, I know nothing; but we must remember, however, that several hypotheses have been made of a more or less experimental nature. Among them

there is one which has been called the principle of *non-heredity*, which postulates that the future of a system depends only on its present state and on its state at an infinitely near moment, or more briefly, that accelerations depend only on positions and velocities. We know that in certain cases this hypothesis is not admissible, at least with the magnitudes directly considered; and sometimes an unjustifiable advantage has been taken of this principle, as, for instance, in the "memory" of matter which recalls its past; and authors have written in terms of deep emotion about the "life" of a fragment of steel. An attempt has been made to construct a theory of these phenomena in which the distant past seems to intervene. Of this I need not speak here. An analyst may think that in cases so complex we must abandon the form of differential equations, and resign ourselves to the consideration of *functional equations*,[1] in which will figure definite integrals bearing witness to an heredity of some sort. When we see the interest which at the present moment attaches to functional equations, one might almost consider it to be due to a presentiment of our future requirements.

VIII.

After having spoken of non-heredity, I scarcely dare touch upon the application of analysis to biology. No doubt it will be some time before we form the functional equations of biological phenomena of a type analogous to those of which I have just spoken.[2] Attempts made so far are not very ambitious. However, we are endeavouring to leave the purely qualitative field and to introduce quantitative measurements. In the question of the variation of certain characters we devote our attention to the statistics of measurements, and represent the results by curves of frequency. The modifications of these curves with successive generations, their decompositions into distinct curves, may give us the measure of the stability of a species or of the rapidity of mutations, and we know the interest that attaches to these questions in recent botanical research (*e.g.* H. de Vries' researches on *Œnanthera*). In all this there are so many parameters that we are led to ask if the infinitesimal method itself may be of any service. Certain laws of a simple arithmetical character, such as those of Mendel, sometimes give us fresh confidence in the old aphorism which I quoted at the beginning of this address, that everything is explained by

[1] I have read M. Freedholm's remarkable paper in the *Acta Mathematica*, t. xxvii., and what was subsequently published in Germany.

[2] In an article on *Lamarck's Principle and the Heredity of Somatic Modifications*, M. Giard thus speaks of heredity: "It is an integral, the sum of the variations produced on each anterior generation by the primary factors of evolution." V. *Controverses transformistes*, p. 135.

numbers; but in spite of our legitimate hopes, it is clear that, as a whole, biology is still far from entering upon a really mathematical period.

This is not the case, according to certain economists, with political economy. After Cournot, the Lausanne school made an extremely interesting effort to introduce mathematical analysis into political economy. Under certain hypotheses, which suit at any rate the limiting cases, we find in learned treatises the equation between the quantities of goods and their prices, reminding us of the equation of virtual velocities in mechanics. This is the equation of economical equilibrium. A function of the quantities plays in this theory an essential part, recalling that of the potential function. Moreover, the most authoritative spokesmen of the school insist on the analogy between economical and mechanical phenomena. "Just as theoretical mechanics considers material points, so," says one of these experts, "pure economy considers the *homo œconomicus*." Naturally, we also find here the analogues of Lagrange's equations, that indispensable matrix of all mechanics. But while we admire these bold investigations, we cannot help feeling that the authors may have neglected certain "hidden masses," as Helmholtz and Hertz would say. But though that may be the case, in these theories there is a curious application of mathematics which, at least in some well circumscribed cases, has already rendered great services.[1]

Gentlemen, I now bring to a close this summary of some of the applications of analysis, and the reflections which have been momentarily suggested by it. It is far from complete; for instance, I have made no mention of the calculus of probabilities, which requires so much subtlety of treatment, the refinements of which Pascal refused to explain to the Chevalier de Méré because he was not a geometer. Its practical utility is of the first rank; its theoretical interest has always been considerable; and it is still greater in the present day, thanks to the importance assumed by the researches which Maxwell called *statistical*, and which tend to exhibit mechanics in quite a new light.[2]

I hope, however, that I have shown in my sketch the origin and the reason of the intimate ties uniting analysis to geometry and physics, and, more generally, to every science bearing upon numerically measurable magnitudes. The reciprocal influence of analysis and physical theories has been in this respect peculiarly instructive. What has the future in store? More difficult problems, corresponding to approximation of a higher order, will bring in complications which we can only vaguely fore-

[1] *v. La méthode mathématique en Économie politique*, by E. Bouvier, and *Petit Traité d'Économie politique mathématique*, by H. Laurent.

[2] Cf. *Elementary Principles in Statistical Mechanics*, by J. Willard Gibbs, and *Leçons sur la théorie des gaz*, by L. Boltzmann.

cast by speaking, as I did just now, of functional equations systematically replacing our present differential equations, or again, of integrations of equations infinite in number and involving an infinity of unknown functions. But even should that come to pass, mathematical analysis will always remain that language which, as Fourier said, "has no symbols to express confused ideas," a language endowed with a wonderful power of transformation, and capable of condensing within its formulas an immense number of results. EMILE PICARD.

REVIEWS.

Kummer's Quartic Surface. By R. W. H. T. HUDSON, M.A., D.Sc. Pp. xii, 222. (Cambridge: at the University Press, 1905.)

The appearance of this book is likely to add, if that be possible, to the general regret aroused by the premature death of the author. The idea of writing it was an excellent one, and has been carried out with marked success, considering the difficulty of choosing and arranging the material. For the study of Kummer's Surface involves a combination of a great variety of analytical methods; groups, configurations, line geometry, non-Euclidean geometry, and theta-functions all play an important part; and in a case of this kind it is not easy to decide how much to take for granted as already familiar to the reader.

The most striking feature of the treatise is the way in which, by a style which is concise without being obscure to a really attentive student, the author has contrived to give the necessary amount of these auxiliary theories so as to be intelligible without reference to other text-books. As an example we may take the last chapter, which deals with Singular Kummer Surfaces. True, this is only a sketch, which requires supplementing by reading the papers of Humbert and others; but it does show very well the *raison d'être* of the transformation of theta-functions, and the effect of the occurrence of singular moduli. And so in other chapters the treatment is broad and suggestive; the main points of the subject are brought out in a way likely to arouse interest, and lead to further research on the part of the reader. This is a pre-eminent merit in a mathematical treatise; and it must have also belonged to the lectures on Kummer's Surface which Hudson gave while preparing his book.

An illustration of Hudson's clearness of thought and expression is given by what he says about geometry of four dimensions. "By 'geometry of four dimensions' is to be understood a method of reasoning about sets of numbers and equations, in which the principles of elementary algebra are clothed in a language analogous to that of ordinary geometry. Although we cannot bring our intuition to bear directly upon four-dimensional configurations, we can do so indirectly by creating an artificial intuition based on analogy." It must be understood, of course, that by "four-dimensional configurations" are meant configurations in a four-dimensional space of points—this is clear enough from the context; ordinary space with lines or spheres as

elements does afford an intuition of a kind, though not analogous in the same immediate way as ordinary solid geometry is to the ordinary geometry of the plane. But with this reservation the passage quoted truly represents the proper way of regarding geometry of four or more dimensions. The power which this "artificial intuition" gives to those who cultivate it is really remarkable; an illustration has been recently given by Mr. Richmond, whose way of obtaining the Kummer (16, 6) configuration is summarised on pp. 129-30 of Hudson's book.

Naturally, Klein's applications of line geometry receive considerable attention. In many respects the fundamental properties of Kummer's Surface, and of its degenerate forms, are closely connected with a family of quadratic complexes; and this is one direction in which further study seems likely to be fruitful. Another is in connection with the application of theta-functions. It is proved by means of them (pp. 184-5; cf. pp. 138-40) that a surface can be found to touch Kummer's Surface all along any given algebraic curve lying thereon, and have no further intersection with the surface. Now this is a purely algebraic theorem, and has no intrinsic connection with theta-functions; it ought, therefore, to be possible to prove it without their aid. Hudson is possibly right in supposing that such a proof would be long and complicated (p. 185); but to provide it would be a real step in advance, and could hardly fail to bring out some characteristic properties of the surface.

No object would be gained by analysing the contents of this treatise in detail: it will be enough to say that, besides the general Kummer Surface, the degenerate forms receive due attention; that conscientious reference is made to the original sources; and that a photograph of a plaster model of the general surface, prefixed to the volume, will greatly help the reader to understand the account given of its geometrical properties. The duty of seeing the later part of the work through the press has been performed by Dr. H. F. Baker and Mr. H. Bateman; a prefatory note by the former contains a brief account of Hudson's career, and a sympathetic appreciation of the merits of this treatise.

G. B. Mathews.

Manual of Quaternions. By Professor C. J. Joly. (Macmillan & Co., 1905.)

Tait has said somewhere that Hamilton first invented the quaternion and then discovered it. The invention consisted in the conception of the three imaginaries *i*, *j*, *k*, with their special laws of combination, and in the construction of an associative algebra with four fundamental units. The quaternion was then discovered to be a complex number representing the ratio of two vectors or directed lines in space. Hamilton was the first who clearly recognised the value of the associative law; and in spite of many imitations his system remains the only tridimensional system of vector analysis governed by this law.

The mathematical world has, however, been slow to recognise the essential merits of Hamilton's calculus. A vast deal of ingenuity and time has been spent by some in finding new notations of the quantities and operators peculiar to quaternions; some have even been beguiled by these notations into a rediscovery of theorems as old as Hamilton's

Lectures. The curious thing is that it does not appear that any really fresh mathematical truth has been brought to light by the inventors and users of vector notations intended to supersede Hamilton's original system. Here, of course, we refer only to tridimensional applications.

It is very refreshing then to open the pages of a book whose author, boldly accepting the form of the calculus as Hamilton developed it, proceeds to unfold its beauties and strength with all the skill of a practised hand. The book begins with a very brief chapter on the addition and subtraction of vectors, a part which necessarily occupied a considerable section of Hamilton's and Tait's treatises. The vector conception has now crept into our elementary books, and in due course will probably become a conspicuous feature even of our most elementary geometries. Graphical methods have within recent years transformed our teaching of algebra; and the vector as a geometrical entity is essentially graphical.

Professor Joly wisely assumes that for students who have made some progress in mathematics the law of vector addition calls for little elaboration. He devotes five pages to it and then plunges into quaternions proper. He takes what he believes to be the "shortest and simplest route." The student he says "cannot be expected to undertake the study of quaternions in the hope of being rewarded by the beauty of the ideas and by the elegance of the analysis. And for his sake, though with reluctance I must confess, I have abandoned Hamilton's methods of establishing the laws of quaternions." Perhaps so, but it is a poor student who despises logical development, beauty of idea, elegance of method, and is content with a "working knowledge of the calculus." We confess to an uneasy feeling that principle has here been sacrificed to expediency. There is a suggestion of "tumbling over the wall" and not coming "in at the Gate which standeth at the beginning of the way." We have often wondered how many students study Clifford's Dynamic for the sake of learning dynamics. Possibly none; and probably as few have *learned* quaternions from that book. Although there is some initial similarity between Professor Joly's method and the tentative nibbling at quaternions which characterises Clifford's book, there is almost immediately a vast divergence. The true quaternion is introduced on page 9, and it dominates the whole treatise. This is as it ought to be. It is possible, as Heaviside has shown, to use effectively much of the notation of quaternions without explicit use of the quaternion itself; but sooner or later the lack of it will be felt. The student should never lose sight of the fact that $S\alpha\beta$ and $V\alpha\beta$ together form by addition a quantity which, however it may operate on or be taken in conjunction with a like quantity, gives rise to a quantity of the same analytical nature. This is the central doctrine of the quaternion calculus.

To give any complete idea of all that Professor Joly's volume contains would be practically to reproduce his table of contents. Beginning with the simpler applications to trigonometry and to the geometry of plane and sphere, he quickly passes into the peculiarities of quaternion differentiation and into the exquisite theory of the linear vector function (or matrix), after which he is ready for all kinds of applications in

geometry of curves and surfaces, and in kinematics and dynamics. The reader cannot fail to be impressed with the directness of the method in all these applications, especially if he is familiar with the ordinary modes of attack. In virtue of this directness of attack and the extraordinary conciseness of notation more detail can be packed into one quaternion page than into three or four pages of ordinary analysis. By what other method, for example, could systematic discussions of line, surface and volume integrals, of spherical harmonics, heterogeneous strain, elastic vibrations, and electro-magnetic theory be given in less than fifty pages? In the variety of the mathematical and physical subjects taken up there are only two other books which can compare with Professor Joly's *Manual*, and these are Hamilton's *Elements* and Tait's *Treatise*.

The greater part of the book is necessarily a development of much that is to be found in the pages of Hamilton, Tait, and M'Aulay; but Professor Joly has a characteristic style of his own, more nearly akin to Hamilton's than to Tait's. In the last two chapters especially are the author's additions more in evidence. These are on Projective Geometry and Hyperspace. The former is based upon a new interpretation of the quaternion; and in the latter Professor Joly gives a sketch of the properties of associative algebras applicable to n-dimensional space.

Professor Joly has certainly succeeded in his aim of providing the student with a *working* book. He takes excursions into many fields of mathematics pure and applied, and the treatment is not superficial. Important applications are worked out in detail; and numerous examples are given by which the student may test his progress. Let the reader accept on trust the initial assumptions and developments, and work earnestly through the succeeding chapters. He will come out in the end a practised quaternionist. C. G. KNOTT.

Leçons sur les fonctions de variables réelles, par E. BOREL; **Leçons sur les fonctions discontinues**, par RENÉ BAIRE; **Le calcul des residus et ses applications à la théorie des fonctions**, par E. LINDELÖF. (Paris, Gauthier-Villars, 1905, 3 f. 50 c. each.)

M. Borel's book is the sixth of his series of monographs on the theory of functions, of which the first appeared as recently as 1898. M. Borel is only human, and by now he has decided to leave to his pupils the work of preparing his lectures for publication. It cannot be said that this method has proved in every case an unqualified success. The first few volumes, prepared by M. Borel himself, and particularly the admirable *Leçons sur les fonctions entières*, were remarkable alike for their originality, for the judgment shewn in the selection of material, and for the lucidity and proportion of the exposition. This high standard has not been maintained in all of the later volumes, some of which have been rather scrappy, and have given the impression of hasty, and at times perfunctory composition. In these respects, however, the present volume is an improvement upon its immediate predecessors, M. Maurice Fréchet having performed his task unusually well. But I cannot help thinking that M. Borel would be fortunate if he could find the time to write his books himself.

The principal problem with which M. Borel deals in this volume is

that of the representation of functions by means of series of polynomials, a form of representation the importance of which was first shown by Weierstrass's well-known theorem that every continuous function of a real variable can be expanded in such a series. M. Borel confines himself to functions of real variables, reserving the complex theory for the next volume of his series; and for the most part he is concerned with continuous functions only, the short chapter on the representation of discontinuous functions containing little more than a discussion of the comparatively simple case in which the aggregate of points of discontinuity is enumerable, and a reference to the results obtained by M. Baire. The consequence is that the book is rather disconnected; for the first two chapters, which deal with the theory of aggregates and continuity and discontinuity in general, contain a good deal which, though very interesting in itself, is really not required for M. Borel's purpose. The central chapters (3 and 4) are excellent,[1] and the long note added by M. Painlevé is perhaps the most interesting feature of the book, though its natural place would not be in this volume, but in the next one.

There is one criticism which will occur to every regular reader of this series, in which M. Borel has now enlisted the collaboration of a number of other eminent mathematicians. There is an amount of repetition which a judicious general editor should be able to diminish; and that the plan of the series is open to this criticism M. Borel, to judge from his remarks in the preface, appears to recognise: 'Il a paru préférable d'admettre parfois quelques brèves redites plutôt que de renoncer à l'indépendance des Volumes de la Collection, chacun d'eux devant pouvoir être lu isolément par un lecteur ayant des connaissances générales d'Analyse. Sans ce principe d'indépendance, on aurait eu tous les inconvénients d'un grand Traité, sans en avoir les avantages.'

In principle, no doubt M. Borel is right. Each author should be asked to deal with some definite question, and he should have full liberty to preface his discussion of it with a general account of those modern developments of analysis which are necessary for his purpose, and with which a reader who has not read the other volumes of the series cannot be expected to be familiar. But he should be very careful to make this general account as short as is consistent with clearness, and to limit it strictly to results which will afterwards be required. This is the course adopted by M. Baire, who, if he is at times a little diffuse, is careful not to encumber his book with unnecessary matter. M. Borel has not set his colleagues so good an example. Why, for example, should he think it necessary to introduce a few pages concerning M. Lebesgue's generalisation of the notion of the definite integral? Most interesting and most important this generalisation certainly is; but it has already been expounded by M. Lebesgue himself in an earlier volume of the series, and no allusion whatever is made to it throughout the remainder of the book.

It will not be necessary to say much about the volume contributed by M. Baire. It is in substance a popular edition of his remarkable

[1] The argument of p. 66 has become inverted in some curious way. The function is '*more* continuous' when $\phi(\epsilon)$ decreases *less* quickly.

memoir, *Sur les fonctions de variables réelles*, published as a thesis in 1899, and afterwards in the *Annali di Matematica*. A good deal of introductory matter has been added, and the argument has been simplified and condensed. The problem of finding the *necessary and sufficient* conditions that a function whose points of discontinuity are given should be capable of representation as the sum of a series of continuous functions is one which most mathematicians would have regarded as hopeless if M. Baire had not completely solved it; and that M. Baire's researches should be made more accessible to the ordinary reader was much to be desired. But it is difficult to resist the impression that these two volumes might well have been condensed into one. M. Baire's results are particularly interesting when applied to differential coefficients of continuous functions. If $f(x)$ is continuous, so is

$$\frac{f(x+h)-f(x)}{h}$$

considered as a function of x. Now, if the differential coefficient exists for every value of x under consideration, and we denote by $h_1, h_2, h_3 \ldots$ a series of positive quantities whose limit is zero,

$$f'(x)=\lim_{n=\infty}\frac{f(x+h_n)-f(x)}{h_n},$$

and is therefore representable as the limit of a sequence of continuous functions, or (what is the same thing) the sum of a series of such functions. M. Baire's results, therefore, give us much important information as regards the possible discontinuities of the differential coefficients of continuous functions.

For M. Lindelöf's *Calcul des Residus* I have nothing but praise. The applications of Cauchy's "calculus" to the theory of functions, and in particular to the summation of series and the theory of analytic continuation, are of the most far-reaching character, and, so far as I know, no one before M. Lindelöf has attempted to give a systematic account of them. Laurent, it is true, published in 1865 a *Théorie des Residus*, which hardly deserved to have been so soon forgotten, but the applications of the theory have multiplied ten times since then. Some new account was urgently necessary, and M. Lindelöf has given us exactly what was wanted. One admirable feature of his book is the thoroughness and exactitude of his historical references, especially to the writings of Cauchy. M. Lindelöf is one of the few mathematicians who have found life long enough to make 'une étude détaillée' of Cauchy's works.

M. Lindelöf very wisely does not trouble himself with all the difficulties as to the *minimum* of assumption required to establish Cauchy's theory, which centre round Goursat's proof of Cauchy's theorem. These difficulties, of course, have absolutely no bearing on the applications with which M. Lindelöf is chiefly concerned. Assuming the continuity of the differential coefficient, he proceeds to show that any analytic function $f(x)$ is itself the differential coefficient of an analytic function $F(x)$ determinate save for an additive constant, a conclusion from which, combined with the definition of the definite

integral along a curvilinear path, Cauchy's theorem immediately follows. There is certainly a great deal to be said for presenting the proof of the theorem in this way.

A short but clear account follows of some familiar applications of the formula

$$f(x)=\frac{1}{2\pi i}\int\frac{f(t)}{t-x}dt$$

and of some others which are not so familiar, such as occur in the proof of Jensen's theorem, the theory of the Bernoullian and Eulerian functions, the factorisation of such functions as $\sin x - ax\cos x$, and the transformation of slowly convergent series. In Chapter III. he comes to the ground which he has made particularly his own. He proves a whole series of general formulae, of which the formula of Plana and Abel,

$$\Sigma f(x) = -\tfrac{1}{2}f(x)+\int^{x} f(x)dx+\frac{1}{i}\int_0^{\infty}\frac{f(x+it)-f(x-it)}{e^{2\pi t}-1}dt,$$

was historically the first. The applications of these formulae to different regions of analysis—Gauss's sums, the Zeta and Gamma functions, the continuation of power series, and the asymptotic behaviour of integral functions—are so numerous that it is impossible to enumerate them here. I will only cite the beautiful formula which defines the behaviour of the function

$$F(x, s)=\frac{x}{1^s}+\frac{x^2}{2^s}+\frac{x^3}{3^s}+\dots$$

near $x=1$, viz.,

$$F(x, s)=\Gamma(1-s)\left(\log\frac{1}{x}\right)^{s-1}+\sum_0^{\infty}\zeta(s-\nu)\frac{(\log x)^{\nu}}{\nu!},$$

which solves completely a problem which has exercised the ingenuity of many mathematicians.

M. Borel's series of monographs is almost indispensable for anyone who is engaged in research in the theory of functions, and with the possible exception of M. Borel's own first two volumes, M. Lindelöf's contribution certainly seems to me the best. G. H. HARDY.

Elements of the Kinematics of a Point and the Rational Mechanics of a Particle. By G. O. JAMES, Ph.D., Instructor in Mathematics, Washington University, St. Louis. New York: Wiley & Sons. London: Chapman & Hall. 1905. Pp. 176. $1.

Presupposing some familiarity with elementary experimental mechanics, and a knowledge of the calculus, the author has written a sketch of the "dynamics of a particle," paying special attention to the effect of the earth's own motion on the motion of a particle on the earth's surface.

The entire absence of examples will appear to many as a defect, but otherwise the book may prove helpful in enabling a student to generalise some of the limited concepts gained by previous experimental work, and to grasp the difficulties of abstract dynamics. C. S. J.

Integral Calculus for Beginners. By ALFRED LODGE, M.A., Mathematical Master at Charterhouse, formerly Professor of Pure Mathematics at the R.I.E. College, Coopers Hill. London: George Bell & Sons. 1905. Pp. 203. 4s. 6d.

This is a companion volume to Professor Lodge's *Differential Calculus for Beginners*, which was reviewed in the *Gazette* [vol. ii., p. 213].

In that volume the student was prepared to practise retracing his steps, and thus, without the use of the integral notation, to perform the operation of integration or anti-differentiation in simple cases. Hence the author is in a position to commence this volume by exhibiting an integral as the limit of a sum; and that no time is wasted in getting to business is evidenced by the fact that the centre of gravity of a parabolic area is worked out at p. 9. The standard methods of integration are clearly explained and illustrated in the first five chapters.

The most novel feature of the book is perhaps the seventh chapter dealing with approximate methods of integration. Here, after the well-known rules of Simpson and Weddle, approximate formulae, recently devised by Mr. R. W. K. Edwards and Professor Lodge himself, are given, for dealing with the case in which the curvilinear boundary of a required area cuts the axis at right angles; a case for which, as is well known, rules of the Simpson type are not well fitted.

Interesting approximate formulae for the elliptic integrals are also given.

A chapter on Moments of Inertia is very welcome, and the book concludes with a chapter on the Gamma functions and with chapters on the differential equations, other than partial, of most frequent occurrence.

The suggestion may be submitted for consideration in a future edition that, while doubtless the theory of Amster's planimeter is too difficult for a first book on the Integral Calculus, yet some of the earlier instruments described in Professor Henrici's British Association report (1894) perform the process of summing up ydx in an obvious manner; and the Integraphs of Professors Boys and Abdank Abakanowicz are also exceedingly interesting concrete embodiments of Integration, viewed as the converse of differentiation.

Professor Lodge's book is likely to maintain the position which his book on the Differential Calculus has won. C. S. JACKSON.

Mechanics. A school course. By W. D. EGGAR, M.A. London: Edward Arnold. 1905. Pp. viii, 288. 3s. 6d.

Mr. Eggar's *Practical Exercises in Geometry* was one of the first of the numerous books which have endeavoured to embody, in a concrete form, the ideas of the "New Geometry."

He has now produced a pioneer book on mechanics, in which simple yet striking experiments and abundant illustrations from familiar objects will compel a student to look on things as they are, and will dissuade him from using terms of art without a notion of their meaning.

The work commences with kinetics, but is so arranged that teachers who prefer to begin with statics can do so. A full account of Galileo's experiments on falling bodies introduces the beginner to a classical example of scientific investigation.

The book is admirably written, and covers more ground than, from its moderate size and attractive print, one would at first imagine; nor has clearness been sacrificed for the sake of brevity, with perhaps one exception in Article 126, dealing with "centrifugal force." Here a diagram, in which the "centrifugal force" is shewn ranking with the real forces, may cause confusion, which the brief statement in the text may not fully remove.

The diagrams are remarkably neat and clear, and there are examples which, as mere algebraical and trigonometrical conundrums have been rigorously excluded, are really examples in mechanics.

On page 13 should not N. 53° E. be E. 53° N.? C. S. JACKSON.

Stereoscopic Views of Solid Geometry Figures. D. C. Heath & Co., Boston, U.S.A.

This set of slides, 93 in number, has been specially designed with reference to the 'Essentials of Solid Geometry' by Prof. W. Wells, a book which has been favourably noticed for the excellence of its text and the diagrams with which it is illustrated. The slides seem to be just what is wanted by a pupil temporarily puzzled as to the position and arrangement of the lines represented by those in the plane diagram. There is a future before the stereoscope in this department of education. Besides the direct use to a beginner of Bk. XI or an equivalent course, we believe that it is likely to act as a powerful incentive to a student to learn enough perspective to enable him to draw his own 'stereographs.'

E. M. LANGLEY.

MATHEMATICAL NOTES.

173. [**K. 13. a.**] The *Remarque Minuscule* (Note 167, *Gazette*, May 1905, p. 176) has also been made by Mr. Steggall (*Proc. Edin. Math. Soc.*, vol. **x**, 1892. N. QUINT ('s-Gravenhage).

174. [**D. 2. a.**]. *The criterion as to a sequence tending to a limit.*

If $s_1, s_2, s_3, \ldots$ be the sequence of real or complex quantities, it does or does not tend to a limit according as it is or is not true that with every positive ϵ, assigned as small as we please, there can be associated a number n such that, for every p, $|s_n - s_{n+p}| < \epsilon$. The ordinary proof that the existence of a limit necessitates this inequality presents no difficulty, but proofs of the converse appear to me needlessly encumbered. The following, in two stages, seems conclusive.

(1) *If the sequence tends to no limit, a positive ϵ' can be taken so small that, whatever quantity s be, and whatever number n be, $|s - s_{n+p}| \nless \epsilon'$ for some p.*

For suppose this is not so. Then with any particular ϵ' goes at least one value of s such that, for some n and every p, $|s - s_{n+p}| < \epsilon'$. A value of s with this property for one ϵ' has it for every greater ϵ'. As ϵ' is diminished, it follows that no new s with the property can arise, but the range of values of s with the property may well diminish. We are supposing, however, that ϵ' cannot be so diminished, remaining assignable and positive, that all values of s cease to have the property. Let s' be one which retains it. Then we are told that, however small ϵ' be assigned, there is always some n such that, for every p, $|s' - s_{n+p}| < \epsilon'$; and this means that the sequence tends to the limit s', so that the supposition made is untenable.

(2) *If, however small ϵ be assigned, there is always some corresponding n such that, for every p, $|s_n - s_{n+p}| < \epsilon$, the sequence tends to a limit.*

For suppose the contrary. Then by (1) there is an assignable ϵ' such that with every value s and every m goes some q such that $|s - s_{m+q}| \nless \epsilon'$. Also our datum assures us that with the ϵ' in question goes some n such that,

for every p, $|s_n - s_{n+p}| < \epsilon'$. In these two inequalities s and m may be chosen at will. Give them the values s_n and n, which are definite. Moreover, p may be chosen at will. Give it the value q, which is made definite by the definite choices of s and m. We thus get two inequalities,

$$|s_n - s_{n+q}| \nless \epsilon' \text{ and } |s_n - s_{n+q}| < \epsilon',$$

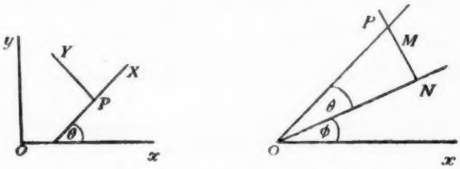
which are inconsistent. We have then supposed an impossibility.

E. B. ELLIOTT.

175. [K. 20. a.]. *The Addition Formulae for Cosine and Sine.*

If PX makes an angle θ with Ox, then the projection of PX on Ox is $PX \cos\theta$. Hence if Oy, PY are obtained from Ox, PX by a turn through a right angle in the positive direction, it follows that the projection of PY on Ox is $PY\cos\left(\theta+\frac{\pi}{2}\right)$, and that the projection of PX on Oy is $PX\cos\left(\theta-\frac{\pi}{2}\right)$.

Now let ON make an angle ϕ with Ox and OP an angle θ with ON. Let NM be the direction obtained from ON by a turn through a right angle in



the positive direction. Let NM cut OP in P and take OP as the unit of length. Then the projection of OP on Ox is $\cos(\theta+\phi)$. But this projection is also the sum of the projections on Ox of ON, NP, that is of $\cos\theta$, $\cos\left(\theta-\frac{\pi}{2}\right)$. Now the projecting factors are $\cos\phi$, $\cos\left(\phi+\frac{\pi}{2}\right)$. Hence we have

$$\cos(\theta+\phi) = \cos\theta\cos\phi + \cos\left(\theta-\frac{\pi}{2}\right)\cos\left(\phi+\frac{\pi}{2}\right)$$
$$= \cos\theta\cos\phi - \sin\theta\sin\phi.$$

The proof applies for angles of any size or sign.

Replacing ϕ by $\phi-\frac{\pi}{2}$, we get the addition formula for sine, and then, in the two addition formulae, replacing ϕ by $-\phi$, we get the formulae for the sine and cosine of $\theta-\phi$.

E. J. NANSON.

176. [D. 6. b.]. *The Fundamental Exponential Limit.*

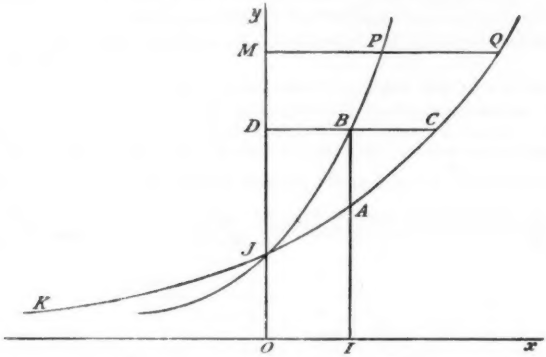
Let the curve KJQ be the graph of $y=a^x$ and let $OI=OJ=1$, so that $IA=a$. In IA or IA produced take any point B where $IB=b$ and through B draw DBC parallel to Ox cutting Oy in D and the graph in C. Then if $DC=c$, we have $b=a^c$, and therefore $b^x=a^{cx}$. Hence if through any point P on the graph JBP of $y=b^x$ MPQ is drawn parallel to Ox cutting Oy in M and the graph of $y=a^x$ in Q, then $MQ=c \, . \, MP$.

Thus from the graph of any one exponential a^x that of any other exponential b^x can be deduced by cutting all the ordinates to y in the proper ratio. Conversely, whatever ratio is used, the graph of some exponential is obtained. By properly choosing the ratio we can therefore make the derived graph have any slope we please at J. There must then be some value of b which gives unit slope at J. Denote this value of b by e, then from the definition of a tangent it follows that

$$\operatorname*{Lt}_{x=0} (e^x - 1)/x = 1.$$

The differential coefficients of e^x, a^x, $\log x$ are at once found, and if

$$(-1)^n y_n = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \ldots + (-1)^n \frac{x^n}{\lfloor n} - e^{-x},$$



we deduce that $dy_n/dx = y_{n-1}$, and hence that $y_1, y_2, \ldots y_n$ are all positive if x is positive, so that e^{-x} lies between s_n and s_{n+1} where

$$s_n = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \ldots + (-1)^n \frac{x^n}{\lfloor n}.$$

Hence as close an approximation as we please can be found for e^{-1}, and therefore also for e. E. J. NANSON.

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The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and published by E. HENDRICKS, M.A., Des Moines, U.S.A.

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Cayley's Mathematical Works. Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor.

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Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

Mathematical Questions and Solutions from the Educational Times. Vol. 18.

Cayley's Collected Mathematical Papers. Vols. VII.-XIII.

BOOKS, ETC., RECEIVED.

Sul movimento di una Sfera che rotola in un piano mobile non orizzontale. By C. ALASIA. pp. 20. (Rivista di Fisica, January, 1905.)

Trasformazioni projettive ad un parametro e loro gruppi continui. By C. ALASIA. pp. 30. (Giornale di Mat. di Battaglini.)

Wiskundig Tijdschrift. Edited by F. J. VAES, CHR. KREDIET, and N. QUINT. Nos. 2-3. 1905. (Blom. Culemborg.)

Nozioni sulla Teoria dei Gruppi di Sostituzioni. By C. ALASIA. pp. 12. (Pitagora. Nos. 8-9. 1905.)

Tables and Constants to Four Figures. Compiled by W. HALL. pp. ix.-60. 3s. net. 1905. (Cam. Univ. Press.)

The Winchester Arithmetic. By C. GODFREY and G. M. BELL. pp. xii., 200. 3s. 1905. (Cam. Univ. Press.)

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Supplemento al Periodico di Matematica. January-June. 1905. Edited by G. LAZZERI. (Giusti, Livorno.)

Beispiel-Sammlung zur Arithmetik und Algebra. By H. SCHUBERT. pp. 147. 80 pf. 1905. (Sammlung Göschen.)

Repetitorium und Aufgabensammlung zur Differentialrechnung. By FR. JUNKER. pp. 128. 80 pf. 1905. (Sammlung Göschen.)

L'Algèbre de la Logique. By L. COUTURAT. No. 24. Scientia. pp. 100. 2 fr. (Gauthier-Villars.)

Le Calcul des Résidus et ses applications à la Théorie des Fonctions. By E. LINDELÖF. pp. 141. 1905. (Gauthier-Villars.)

Vorlesungen über die Vektorenrechnung. By E. JAHNKE. pp. xii., 235. 5.60 m. 1905. (Teubner.)

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E. R. Neumann. Studien über die Methoden von C. Neumann und G. Robin zur Lösing der beiden Randwertaufgaben der Potentialtheorie. pp. xxiv., 194. 10 m. 1905. (Teubner.)

The Intersections of plane curves, with extensions to n-dimensional algebraic manifolds. By F. S. MACAULAY, D.Sc. pp. 284-312 (from the Proceedings of the Third International Mathematic Congress, Heidelberg, 1904). (Teubner.)

A Course in Practical Mathematics. By F. M. SAXELBY. pp. viii., 438. 6s. 6d. 1905. (Longmans, Green.)

An Elementary Treatise on the Calculus for Engineering Students. By J. GRAHAM. 3rd edition. pp. xii., 276. 7s. 6d. 1905. (Spon.)

Mathematical Recreations and Essays. By W. W. ROUSE BALL. 4th edition. pp. xvi., 388. 7s. net. 1905. (Macmillan.)

The Primary Arithmetic. Edited by W. BRIGGS. Part I. 6d. 1905. (Univ. Tutorial Series.)

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Examples in Arithmetic, with some notes on method. By C. O. TUCKEY. pp. 215, xii., xxxix. 3s. 1905. (Bell.)

On the Theory of Connexes. By PROF. D. SINTSOFF. pp. 72. 1904. (Russian.) [Kharkoff.]

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On the real elements of certain Classes of Geometrical Configurations. L. E. DICKSON. *The Continuum as a Type of Order.* E. V. HUNTINGTON. *On Integrating Factors.* P. SAUREL. *Concerning Series of Analytic Functions.* M. B. PORTER.

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